Short probabilistic proof of the Brascamp-Lieb and Barthe theorems

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1 Introduction

A Brascamp-Lieb datum on $\mathbb{R}^n$ is a finite sequence

$$(c_1, B_1), \dots, (c_m, B_m) \tag{1}$$

where c_i is a positive number and $B_i \colon \mathbb{R}^n \to \mathbb{R}^{n_i}$ is linear and onto. The Brascamp-Lieb constant associated to this datum is the smallest real number C such that the inequality

$$\int_{\mathbb{R}^n} \prod_{i=1}^m (f_i \circ B_i)^{c_i} \,\mathrm{d}x \leq C \prod_{i=1}^m \Bigl(\int_{\mathbb{R}^{n_i}} f_i \,\mathrm{d}x \Bigr)^{c_i} \tag{2}$$

holds for every set of non-negative integrable functions $f_i \colon \mathbb{R}^{n_i} \to \mathbb{R}$. The Brascamp-Lieb theorem [8, 12] asserts that (2) is saturated by Gaussian functions. In other words if (2) holds for every functions $f_1, \dots, f_m$ of the form

$$f_i(x) = \mathrm{e}^{-\langle A_i x, x \rangle / 2}$$

where A_i is a symmetric positive definite matrix on $\mathbb{R}^{n_i}$ then (2) holds for every set of functions $f_1, \dots, f_m$.

The reversed Brascamp-Lieb constant associated to (1) is the smallest constant C_r such that for every non-negative measurable functions $f_1, \dots, f_m, f$ satisfying

$$\prod_{i=1}^m f_i(x_i)^{c_i} \leq f\Bigl(\sum_{i=1}^m c_i B_i^* x_i \Bigr) \tag{3}$$

for every $(x_1, \dots, x_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ we have

$$\prod_{i=1}^m \Bigl(\int_{\mathbb{R}^{n_i}} f_i \,\mathrm{d}x \Bigr)^{c_i} \leq C_r \int_{\mathbb{R}^n} f \,\mathrm{d}x. \tag{4}$$

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It was shown by Barthe [1] that again Gaussian functions saturate the inequality. The original paper of Brascamp and Lieb [8] rely on symmetrization techniques. Barthe's argument uses optimal transport and works for both the direct and the reversed inequality. More recent proofs of the direct inequality [4, 5, 9, 10] all rely on semi-group techniques. Barthe and Huet [2] have a semi-group argument that works for both the direct and reversed inequality, provided the Brascamp-Lieb datum satisfies

$$\begin{aligned} & B_i B_i^* = \mathrm{id}_{\mathbb{R}^{n_i}}, \ \forall i \leq m, \\ & \sum_{i=1}^m c_i B_i^* B_i = \mathrm{id}_{\mathbb{R}^n}. \end{aligned} \tag{5}$$

This constraint is called the *frame condition* hereafter.
The purpose of this article is to give a short probabilistic proof of the Brascamp-Lieb and Barthe theorems. Our main tool shall be a representation formula for the quantity

$$\ln\Big(\int \mathrm{e}^{g(x)} \ \gamma(\mathrm{d}x) \Big),$$

where γ is a Gaussian measure. Let us describe it briefly. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and let

$$(W_t)_{t\in[0,T]}$$

be a Brownian motion taking values in $\mathbb{R}^n$ (we fix a finite time horizon T). Assuming that the covariance matrix A of W (i.e. the covariance matrix of the random vector W_1) has full rank, we let $\mathbb{H}$ be the associated Cameron-Martin space; namely the Hilbert space of absolutely continuous paths $u\colon [0,T] \to \mathbb{R}^n$ starting from 0, equipped with the norm

$$\|u\|_{\mathbb{H}} = \Big(\int_0^T \langle A^{-1} \dot{u}_s, \dot{u}_s \rangle \, \mathrm{d}s \Big)^{1/2}.$$

In the sequel we call *drift* any adapted process U which belongs to $\mathbb{H}$ almost surely. The following formula is due to Boué and Dupuis [7] (see also [6, 11]).

Proposition 1. *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be measurable and bounded from below, then*

$$\log\Big(\mathsf{E}\mathrm{e}^{g(W_T)} \Big) = \sup \Big[\mathsf{E}\Big(g(W_T + U_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \Big) \Big]$$

where the supremum is taken over all drifts U.

In [6], Borell rediscovers this formula and shows that it yields the Prékopa-Leindler inequality (a reversed form of Hölder's inequality) very easily. Later on Cordero and Maurey noticed that under the frame condition, both the direct and reversed Brascamp-Lieb inequalities could be recovered this way (this was not published but is explained in [11]). The purpose of this article is, following Borell, Cordero and Maurey, to show that the Brascamp-Lieb and Barthe theorems in full generality are direct consequences of Proposition 1.

2 The direct inequality

Replace f_i by $x \mapsto f_i(x/\lambda)$ in inequality (2). The left-hand side of the inequality is multiplied by λ^n and the right-hand side by $\lambda^{\sum_{i=1}^m c_i n_i}$. Therefore, a necessary condition for C to be finite is

$$\sum_{i=1}^m c_i n_i = n.$$

This homogeneity condition will be assumed throughout the rest of the article.

Theorem 2. *Assume that there exists a matrix A satisfying*

$$A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i. \tag{6}$$

Then the Brascamp-Lieb constant is

$$C = \Big(\frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \Big)^{1/2},$$

and there is equality in (2) *for the following Gaussian functions*

$$f_i \colon x \in \mathbb{R}^{n_i} \mapsto \mathrm{e}^{-\langle (B_i A B_i^*)^{-1} x, x\rangle/2}, \quad i \leq m. \tag{7}$$

Remark. If the frame condition (5) holds then $A = \mathrm{id}_{\mathbb{R}^n}$ satisfies (6) and the Brascamp-Lieb constant is 1.

Proof. Because of (6), if the functions f_i are defined by (7) then

$$\prod_{i=1}^m \big(f_i(B_i x) \big)^{c_i} = \mathrm{e}^{-\langle A^{-1} x, x \rangle/2}.$$

The equality case follows easily (recall the homogeneity condition $\sum c_i n_i = n$). Let us prove the inequality. Let $f_1, \dots, f_m$ be non-negative integrable functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively and let

$$f \colon x \in \mathbb{R}^n \mapsto \prod_{i=1}^m f_i(B_i x)^{c_i}.$$

Fix $\delta > 0$, let $g_i = \log(f_i + \delta)$ for every $i \leq m$ and let

$$g(x) = \sum_{i=1}^m c_i g_i(B_i x).$$

The functions $(g_i)_{i \leq m}, g$ are bounded from below. Fix a time horizon T, let $(W_t)_{t \geq T}$ be a Brownian motion on $\mathbb{R}^n$, starting from 0 and having covariance

A; and let $\mathbb{H}$ be the associated Cameron-Martin space. By Proposition 1, given $\epsilon > 0$, there exists a drift U such that

$$\begin{aligned}\log\Big(\mathsf{E}\mathrm{e}^{g(W_T)}\Big) &\leq \mathsf{E}\Big(g(W_T+U_T) - \frac{1}{2}\|U\|_{\mathbb{H}}^2\Big) + \epsilon \\ &= \sum_{i=1}^m c_i \mathsf{E} g_i(B_iW_T + B_iU_T) - \frac{1}{2}\mathsf{E}\|U\|_{\mathbb{H}}^2 + \epsilon.\end{aligned} \tag{8}$$

The process B_iW is a Brownian motion on $\mathbb{R}^{n_i}$ with covariance $B_iAB_i^*$. Set $A_i = B_iAB_i^*$ and let $\mathbb{H}_i$ be the Cameron-Martin space associated to B_iW. Equality (6) gives

$$\langle A^{-1}x, x\rangle = \sum_{i=1}^m c_i \langle A_i^{-1}B_ix, B_ix\rangle$$

for every $x \in \mathbb{R}^n$. This implies that

$$\|u\|_{\mathbb{H}}^2 = \sum_{i=1}^m c_i \|B_iu\|_{\mathbb{H}_i}^2$$

for every absolutely continuous path $u\colon [0,T] \to \mathbb{R}^n$. So that (8) becomes

$$\log\Big(\mathsf{E}\mathrm{e}^{g(W_T)}\Big) \leq \sum_{i=1}^m c_i \mathsf{E}\Big(g_i(B_iW_T + B_iU_T) - \frac{1}{2}\|B_iU\|_{\mathbb{H}_i}^2\Big) + \epsilon.$$

By Proposition 1 again we have

$$\mathsf{E}\Big(g_i(B_iW_T + B_iU_T) - \frac{1}{2}\|B_iU\|_{\mathbb{H}_i}^2\Big) \leq \log\Big(\mathsf{E}\mathrm{e}^{g_i(B_iW_T)}\Big)$$

for every $i \leq m$. We obtain (dropping ϵ which is arbitrary)

$$\log\Big(\mathsf{E}\mathrm{e}^{g(W_T)}\Big) \leq \sum_{i=1}^m c_i \log\Big(\mathsf{E}\mathrm{e}^{g_i(B_iW_T)}\Big). \tag{9}$$

Recall that $f \leq \mathrm{e}^g$ and observe that

$$\prod_{i=1}^m \Big(\mathsf{E}(\mathrm{e}^{g_i}(B_iW_T)\Big)^{c_i} \leq \prod_{i=1}^m \Big(\mathsf{E}f_i(B_iW_T)\Big)^{c_i} + O(\delta^c),$$

for some positive constant c. Inequality (9) becomes (dropping the $O(\delta^c)$ term)

$$\mathsf{E}f(W_T) \leq \prod_{i=1}^m \Big(\mathsf{E}f_i(B_iW_T)\Big)^{c_i}. \tag{10}$$

Since W_T is a centered Gaussian vector with covariance TA

$$\mathsf{E}f(W_T) = \frac{1}{(2\pi T)^{n/2}\det(A)^{1/2}} \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\langle A^{-1}x,x\rangle/2T}\,\mathrm{d}x,$$

and there a similar equality for $\mathsf{E}f_i(B_iW_T)$. Then it is easy to see that letting T tend to $+\infty$ in inequality (10) yields the result (recall that $\sum c_in_i = n$). $\square$

Example (Optimal constant in Young's inequality). Young's convolution inequality asserts that if $p, q, r \geq 1$ and are linked by the equation

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{11}$$

then

$$\|F * G\|_r \leq \|F\|_p \|G\|_q,$$

for all $F \in L_p$ and $G \in L_q$. When either p, q or r equals 1 or $+\infty$ the inequality is a consequence of Hölder's inequality and is easily seen to be sharp. On the other hand when p, q, r belong to the open interval $(1, +\infty)$ the best constant C in the inequality

$$\|F * G\|_r \leq C \|F\|_p \|G\|_q,$$

is actually smaller than 1. Let us compute it using the previous theorem. Observe that by duality C is the best constant in the inequality

$$\int_{\mathbb{R}^2} f^{c_1}(x+y) g^{c_2}(y) h^{c_3}(x) \, \mathrm{d}x \mathrm{d}y \leq C \Big(\int_{\mathbb{R}} f \Big)^{c_1} \Big(\int_{\mathbb{R}} g \Big)^{c_2} \Big(\int_{\mathbb{R}} h \Big)^{c_3}, \tag{12}$$

where

$$c_1 = \frac{1}{p}, \; c_2 = \frac{1}{q}, \; c_3 = 1 - \frac{1}{r}.$$

In other words C is the Brascamp-Lieb constant in $\mathbb{R}^2$ associated to the data

$$(c_1, B_1), (c_2, B_2), (c_3, B_3),$$

where $B_1 = (1, 1)$, $B_2 = (0, 1)$ and $B_3 = (1, 0)$. According to the previous result, we have to find a positive definite matrix A satisfying

$$A^{-1} = \sum_{i=1}^{3} c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

Letting $A = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$, this equation turns out to be equivalent to

$$\begin{aligned} (1 - c_2) xy + yz + c_2 z^2 &= 0 \\ (1 - c_3) xy + xz + c_3 z^2 &= 0 \\ c_1 + c_2 + c_3 &= 2. \end{aligned}$$

The third equation is just the Young constraint (11). The first two equations admit two families of solutions: either (x, y, z) is a multiple of $(1, 1, -1)$ or (x, y, z) is a multiple of

$$\big(c_3(1 - c_3), c_2(1 - c_2), -(1 - c_2)(1 - c_3) \big).$$

The constraint $xy - z^2 > 0$ rules out the first solution. The second solution is fine since c_1, c_2 and c_3 are assumed to belong to the open interval $(0,1)$. By Theorem 2, the best constant in (12) is

$$C = \Big(\frac{\det(A)}{\prod_{i=1}^3 \det(B_i A B_i^*)^{c_i}}\Big)^{1/2} = \Big(\frac{(1-c_1)^{1-c_1}(1-c_2)^{1-c_2}(1-c_3)^{1-c_3}}{c_1^{c_1} c_2^{c_2} c_3^{c_3}}\Big)^{1/2}.$$

In terms of p, q, r we have

$$C = \Big(\frac{p^{1/p}\ q^{1/q}\ r'^{1/r'}}{p'^{1/p'}\ q'^{1/q'}\ r^{1/r}}\Big)^{1/2}$$

where p', q', r' are the conjugate exponents of p, q, r, respectively. This is indeed the best constant in Young's inequality, first obtained by Beckner [3].

3 The reversed inequality

Theorem 3. *Again, assume that there is a matrix A satisfying* (6)*. Then the reversed Brascamp-Lieb constant is*

$$C_r = \Big(\frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}}\Big)^{1/2}.$$

There is equality in (4) *for the following Gaussian functions*

$$\begin{aligned} f_i &: x \in \mathbb{R}^{n_i} \mapsto \mathrm{e}^{-\langle B_i A B_i^* x, x\rangle/2}, \quad i \le m. \\ f &: x \in \mathbb{R}^n \mapsto \mathrm{e}^{-\langle Ax, x\rangle/2}. \end{aligned}$$

Remark. Observe that under condition (6) the Brascamp-Lieb constant and the reversed constant are the same, but the extremizers differ.

We shall use the following elementary lemma.

Lemma 4. *Let $A_1, \ldots, A_m$ be positive definite matrices on $\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_m}$, respectively and let*

$$A = \Big(\sum_{i=1}^m c_i B_i^* A_i^{-1} B_i\Big)^{-1}.$$

Then for all $x \in \mathbb{R}^n$

$$\langle Ax, x\rangle = \inf\Big\{\sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle,\ \sum_{i=1}^m c_i B_i^* x_i = x\Big\}.$$

Proof. Let $x_1, \ldots, x_m$ and let

$$x = \sum_{i=1}^m c_i B_i^* x_i. \tag{13}$$

Then by the Cauchy-Schwarz inequality (recall that the matrices A_i are positive definite)

$$\begin{aligned}\langle Ax, x\rangle &= \sum_{i=1}^m c_i \langle Ax, B_i^* x_i\rangle = \sum_{i=1}^m c_i \langle B_i A x, x_i\rangle \\ &\le \Big(\sum_{i=1}^m c_i \langle A_i^{-1} B_i A x, B_i A x\rangle\Big)^{1/2} \Big(\sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle\Big)^{1/2} \\ &= \langle Ax, x\rangle^{1/2} \, \Big(\sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle\Big)^{1/2}.\end{aligned}$$

Besides, given $x \in \mathbb{R}^n$, set $x_i = A_i^{-1} B_i A x$ for all $i \le m$. Then (13) holds and there is equality in the above Cauchy-Schwarz inequality. This concludes the proof. $\square$

Proof of Theorem 3. The equality case is a straightforward consequence of the hypotethis (6) and Lemma 4, details are left to the reader.
Let us prove the inequality. There is no loss of generality assuming that the functions $f_1, \dots, f_m$ are bounded from above (otherwise replace f_i by $\max(f_i, k)$, let k tend to $+\infty$ and use monotone convergence). Fix $\delta > 0$ and let $g_i = \log(f_i + \delta)$ for every $i \le m$. By (3) and since the functions f_i are bounded from above, there exist positive constants c, C such that the function

$$g \colon x \in \mathbb{R}^n \mapsto \log\big(f(x) + C\delta^c\big),$$

satisfies

$$\sum_{i=1}^m c_i g_i(x_i) \le g\Big(\sum_{i=1}^m c_i B_i^* x_i\Big) \tag{14}$$

for every $x_1, \dots, x_m$. Observe that the functions $(g_i)_{i\le m}, g$ are bounded from below. Let $(W_t)_{t\le T}$ be a Brownian motion on $\mathbb{R}^n$ having covariance matrix A. Set $A_i = B_i A B_i^*$, then $A_i^{-1} B_i W$ is a Brownian motion on $\mathbb{R}^{n_i}$ with covariance matrix

$$(A_i^{-1} B_i) A (A_i^{-1} B_i)^* = A_i^{-1} (B_i A B_i^*) A_i^{-1} = A_i^{-1}.$$

Let $\mathbb{H}_i$ be the associated Cameron-Martin space. By Proposition 1 there exists a ($\mathbb{R}^{n_i}$-valued) drift U_i such that

$$\log\Big(\mathsf{E} e^{g_i(A_i^{-1} B_i W_T)}\Big) \le \mathsf{E}\Big(g_i(A_i^{-1} B_i W_T + (U_i)_T) - \frac{1}{2}\|U_i\|_{\mathbb{H}_i}^2\Big) + \epsilon. \tag{15}$$

By (14) and (6)

$$\begin{aligned}\sum_{i=1}^m c_i g_i(A_i^{-1} B_i W_T + (U_i)_T) &\le g\Big(\sum_{i=1}^m c_i B_i^* (A_i^{-1} B_i W_T + (U_i)_T)\Big) \\ &= g\Big(A^{-1} W_T + \sum_{i=1}^m c_i B_i^* (U_i)_T\Big).\end{aligned}$$

The Brownian motion $(A^{-1}W)_{t\le T}$ has covariance matrix $A^{-1}A(A^{-1})^* = A^{-1}$. Let $\mathbb{H}$ be the associated Cameron-Martin space. Lemma 4 shows that

$$\Big\langle A\Big(\sum_{i=1}^m c_i B_i^* x_i\Big), \sum_{i=1}^m c_i B_i^* x_i\Big\rangle \le \sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle$$

for every $x_1,\dots,x_m$ in $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_m}$, respectively. Therefore

$$\Big\|\sum_{i=1}^m c_i B_i^* u_i\Big\|_{\mathbb{H}}^2 \le \sum_{i=1}^m c_i \|u_i\|_{\mathbb{H}_i}^2.$$

for every sequence of absolutely continuous paths $(u_i\colon [0,T]\to\mathbb{R}^{n_i})_{i\le m}$. Thus multiplying (15) by c_i and summing over i yields

$$\sum_{i=1}^m c_i \log\Big(\mathsf{E}\mathrm{e}^{g_i(A_i^{-1}B_iW_T)}\Big)$$
$$\le \mathsf{E}\Big[g\big(A^{-1}W_T + \sum_{i=1}^m c_i B_i^*(U_i)_T\big) - \frac{1}{2}\big\|\sum_{i=1}^m c_i B_i^* U_i\big\|_{\mathbb{H}}^2\Big] + \sum_{i=1}^m c_i\epsilon.$$

Hence, using Proposition 1 again and dropping ϵ again,

$$\sum_{i=1}^m c_i \log\Big(\mathsf{E}\mathrm{e}^{g_i(A_i^{-1}B_iW_T)}\Big)^{c_i} \le \log\Big(\mathsf{E}\mathrm{e}^{g(A^{-1}W_T)}\Big). \tag{16}$$

Recall that $f_i \le \mathrm{e}^{g_i}$ for every $i\le m$ and that $\mathrm{e}^g = f + C\delta^c$. Since δ is arbitrary, inequality (16) becomes

$$\prod_{i=1}^m \Big(\mathsf{E}f_i(A_i^{-1}B_iW_T)\Big)^{c_i} \le \mathsf{E}f(A^{-1}W_T).$$

Again, letting T tend to $+\infty$ in this inequality yields the result. □

4 The Brascamp-Lieb and Barthe theorems

So far we have seen that both the direct inequality and the reversed version are saturated by Gaussian functions when there exists a matrix A such that

$$A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i. \tag{17}$$

In this section, we briefly explain why this yields the Brascamp-Lieb and Barthe theorems.
Applying (2) to Gaussian functions gives

$$\prod_{i=1}^m \det(A_i)^{c_i} \le C^2 \det\big(\sum_{i=1}^m c_i B_i^* A_i B_i\big), \tag{18}$$

for every sequence $A_1, \dots, A_m$ of positive definite matrices on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$. Let C_g be the Gaussian Brascamp-Lieb constant; namely the best constant in the previous inequality. We have $C_g \leq C$ and it turns out that applying (4) to Gaussian functions yields $C_g \leq C_r$ (one has to apply Lemma 4 at some point). It is known since the work of Carlen and Cordero [9] that there is a dual formulation of (2) in terms of relative entropy. In the same way, there is a dual formulation of (18). For every positive matrix A on $\mathbb{R}^n$, one has

$$\log\det(A) = \inf_{B>0}\big(\operatorname{tr}(AB) - n - \log(\det(B))\big),$$

with equality when $B = A^{-1}$. Using this and the equality $\sum_{i=1}^m c_i n_i = n$, it is easily seen that C_g is also the best constant such that the inequality

$$\det(A) \leq C_g^2 \prod_{i=1}^m \det(B_i A B_i^*)^{c_i} \tag{19}$$

holds for every positive definite matrix A on $\mathbb{R}^n$.

Example. Assume that $m = n$, that $c_1 = \cdots = c_n = 1$ and that $B_i(x) = x_i$ for $i \in [n]$. Inequality (18) trivially holds with constant 1 (and there is equality for every $A_1, \dots, A_n$). On the other hand (19) becomes

$$\det(A) \leq \prod_{i=1}^n a_{ii},$$

for every positive definite A, with equality when A is diagonal. This is Hadamard's inequality.

Lemma 5. *If A is extremal in* (19) *then A satisfies* (17).

Proof. Just compute the gradient of the map

$$A > 0 \mapsto \log\det(A) - \sum_{i=1}^m c_i \log\det(B_i A B_i^*). \qquad \square$$

Therefore, if the constant C_g is finite and if there is an extremizer A in (19) then A satisfies (17) and together with the results of the previous sections we get the Brascamp-Lieb and Barthe equalities

$$C = C_r = C_g. \tag{20}$$

Although it may happen that $C_g < +\infty$ and no Gaussian extremizer exists, there is a way to bypass this issue. For the Brascamp-Lieb theorem, there is an abstract argument showing that is it is enough to prove the equality $C = C_g$ when there is a Gaussian extremizer. This argument relies on:

1. A criterion for having a Gaussian extremizer, due to Barthe [1] in the rank 1 case (namely when the dimensions n_i are all equal to 1) and Bennett, Carbery, Christ and Tao [5] in the general case.

2. A multiplicativity property of C and C_g due to Carlen, Lieb and Loss [10] in the rank 1 case and BCCT again in general.

There is no point repeating this argument here, and we refer to [10, 5] instead. This settles the case of the $C = C_g$ equality. As for the $C = C_r$ equality, we observe that the above argument can be carried out verbatim once the mutliplicativity property of the reversed Brascamp-Lieb constant is established. This is the purpose of the rest of the article.

Definition 6. Given a proper subspace E of $\mathbb{R}^n$ we let $B_{i,E}$ be the restriction of B_i to E and

$$B_{i,E^\perp} : x \in E^\perp \mapsto q_i \circ B_i x,$$

where q_i is the orthogonal projection onto $(B_i E)^\perp$. Let $C_{r,E}$ be the reversed Brascamp-Lieb constant on E associated to the datum

$$(c_1, B_{1,E}), \ldots, (c_m, B_{m,E})$$

and $C_{r,E^\perp}$ be the Brascamp-Lieb constant on $E^\perp$ associated to the datum

$$(c_1, B_{1,E^\perp}), \ldots, (c_m, B_{m,E^\perp})$$

Remark. It may happen that the restriction of B_i to E is identically 0. In the sequel, we take the convention that a Brascamp-Lieb datum is allowed to contain maps B_i which are identically 0, but that these are discarded for the computation of the associated Brascamp-Lieb constants.

Proposition 7. *Let E be a proper subspace of $\mathbb{R}^n$, and assume that E is critical, in the sense that*

$$\dim(E) = \sum_{i=1}^m c_i \dim(B_i E).$$

Then $C_r = C_{r,E} \times C_{r,E^\perp}$.

Bennett, Carbery, Christ and Tao proved the corresponding property of C and C_g, we adapt their argument to prove the multiplicativity of C_r. This adaptation is straightforward for the inequality

$$C_r \leq C_{r,E} \times C_{r,E^\perp}$$

and is left to the reader (observe that criticality of E is not even needed). We start the proof of the reversed inequality with a couple of simple observations.

Lemma 8. *Upper semi-continuous functions having compact support saturate the reversed Brascamp-Lieb inequality.*

Proof. The regularity of the Lebesgue measure implies that given a non-negative integrable function f_i on $\mathbb{R}^{n_i}$ and $\epsilon > 0$ there exists a non-negative linear combination of indicators of compact sets g_i satisfying

$$g_i \leq f_i \quad \text{and} \quad \int_{\mathbb{R}^{n_i}} f_i \,\mathrm{d}x \leq (1+\epsilon) \int_{\mathbb{R}^{n_i}} g_i \,\mathrm{d}x.$$

The lemma follows easily. □

The proof of the following lemma is left to the reader.

Lemma 9. *If $f_1, \dots, f_m$ are upper semi-continuous functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$ respectively, then the function f defined on $\mathbb{R}^n$ by*

$$f(x) = \sup\Big(\prod_{i=1}^m f_i(x_i)^{c_i}, \ \sum_{i=1}^m c_i B_i^* x_i = x \Big),$$

is upper semi-continuous as well.

Remark. If the Brascamp-Lieb datum happens to be *degenerate*, in the sense that the map $(x_1, \dots, x_m) \mapsto \sum_{i=1}^m B_i^* x_i$ is not onto, then Brascamp-Lieb constants are easily seen to be $+\infty$. Still the previous lemma remains valid, provided the convention $\sup(\emptyset) = 0$ is adopted.

Let us prove that $C_{r,E} \times C_{r,E^\perp} \leq C_r$. By Lemma 8, it is enough to prove that the inequality

$$\prod_{i=1}^m \Big(\int_{B_i E} f_i \, \mathrm{d}x \Big)^{c_i} \times \prod_{i=1}^m \Big(\int_{(B_i E)^\perp} g_i \, \mathrm{d}x \Big)^{c_i} \leq C_r \Big(\int_E f \, \mathrm{d}x \Big) \Big(\int_{E^\perp} g \, \mathrm{d}x \Big).$$

holds for every compactly supported upper semi-continuous functions $(f_i)_{i \leq m}$ and $(g_i)_{i \leq m}$, where f and g are defined by

$$\begin{aligned} f \colon x \in E &\mapsto \sup\Big(\prod_{i=1}^m f_i(x_i)^{c_i}, \ \sum_{i=1}^m c_i (B_{i,E})^* x_i = x \Big) \\ g \colon y \in E^\perp &\mapsto \sup\Big(\prod_{i=1}^m g_i(y_i)^{c_i}, \ \sum_{i=1}^m c_i (B_{i,E^\perp})^* x_i = y \Big). \end{aligned}$$

Let $\epsilon > 0$. For $i \leq m$ define a function h_i on $\mathbb{R}^{n_i}$ by

$$h_i(x + y) = f_i(x/\epsilon) g_i(y), \quad \forall x \in B_i E, \, \forall y \in (B_i E)^\perp,$$

and let

$$h \colon z \in \mathbb{R}^n \mapsto \sup\Big(\prod_{i=1}^m h_i(z_i)^{c_i}, \ \sum_{i=1}^m c_i B_i^* z_i = z \Big).$$

By definition of the reversed Brascamp-Lieb constant C_r

$$\prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} h_i \, \mathrm{d}x \Big)^{c_i} \leq C_r \int_{\mathbb{R}^n} h \, \mathrm{d}x. \tag{21}$$

Using the equality $\sum_{i=1}^m c_i \dim(B_i E) = \dim(E)$ we get

$$\epsilon^{-\dim(E)} \prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} h_i \, \mathrm{d}x \Big)^{c_i} = \prod_{i=1}^m \Big(\int_{B_i E} f_i \, \mathrm{d}x \Big)^{c_i} \times \prod_{i=1}^m \Big(\int_{(B_i E)^\perp} g_i \, \mathrm{d}x \Big)^{c_i}.$$

On the other hand, we let the reader check that for every $x \in E, y \in E^\perp$

$$h(\epsilon x + y) \leq f(x) g_\epsilon(y),$$

where

$$g_\epsilon(y) = \sup\bigl(g(y'),\ |y - y'| \leq K\epsilon\bigr)$$

and K is a constant depending on the diameters of the supports of the functions f_i. Therefore

$$\epsilon^{-\dim E} \int_{\mathbb{R}^n} h \,\mathrm{d}x = \int_{E\times E^\perp} h(\epsilon x + y)\,\mathrm{d}x\mathrm{d}y \leq \Bigl(\int_E f \,\mathrm{d}x\Bigr)\Bigl(\int_{E^\perp} g_\epsilon \,\mathrm{d}x\Bigr).$$

Inequality (21) becomes

$$\prod_{i=1}^m \Bigl(\int_{B_i E} f_i \,\mathrm{d}x\Bigr)^{c_i} \times \prod_{i=1}^m \Bigl(\int_{(B_i E)^\perp} g_i \,\mathrm{d}x\Bigr)^{c_i} \leq C_r \Bigl(\int_E f \,\mathrm{d}x\Bigr)\Bigl(\int_{E^\perp} g_\epsilon \,\mathrm{d}x\Bigr).$$

Clearly g has compact support, and g is upper semi-continous by Lemma 9. This implies easily that

$$\lim_{\epsilon\to 0} \int_{E^\perp} g_\epsilon \,\mathrm{d}x = \int_{E^\perp} g \,\mathrm{d}x,$$

which concludes the proof.

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